New Multi Step Methods for Solving General Mixed inverse Quasi Variational Inequalities

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ABSTRACT

Some new classes of general mixed inverse quasi variational inclusions are introduced and studied. Several important special cases are highlighted as applications of these problems. The auxiliary principle technique is exploited to suggest some new multi step iterative methods for solving general mixed inverse quasi variational inequalities. We prove that the convergence of the proposed methods only requires the partially relaxed strongly monotonicity of the operator and skew symmetric bifunction, which is weaker than co-coercivity. As special cases, we obtain various known and new results for solving various classes of variational inequalities and related problems. These multi step methods include Mann iteration, Ishikawa (two-step) and Noor (three-step) iterations as special cases..

KEYWORDS

Inverse Quasi variational inequalities, auxiliary principle, multi step iterative methods, convergence, fixed points.

1. Introduction

Variational inequalities theory provides us with a simple, natural, unified, novel and general framework to study a wide class of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, transportation, finance, economics, and optimization. For more details, see [1, 3, 5–9, 11–14, 17–20, 22–43, 45–56]. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. Variational inequalities have been generalized and extended in different directions using the novel and innovative techniques. An important and useful generalization of variational inequalities is called the general mixed quasi variational inequality involving the nonlinear bifunction. If the nonlinear bifunction involving the general mixed variational inequalities is a proper, convex and lower-semicontinuous with respect to the first argument, then it has been shown [28–33, 35, 45] that the general mixed quasi variational inequalities are equivalent to the fixed point and resolvent equations. This alternative formulation was used to introduce three-step iterations to consider the existence of a solution and iterative

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schemes for solving the quasi variational inclusions. These three-step iterations are known Noor iterations, which contains Picard method, Mann(one step)iteration and Ishikawa (two-step) iterations as special cases. Suantai et. al. [51] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and image in painting. Noor (three step) iterations have influenced the research in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design, see [2–4, 21, 40, 49, 51, 55] In this approach, one has to evaluate the resolvent operator, which is itself a difficult problem. Secondly resolvent method, descent methods and their variant forms cannot be extended and modified to suggest iterative methods for solving the mixed quasi variational inequalities involving the nonlinear bifunction.

To overcome these drawbacks, the auxiliary principle technique has been developed, the origin of which can be traced back to Lions and Stampacchia [20]. Glowinski et al.[14] used this technique to study the existence of a solution of the mixed variational inequalities. In recent years, Noor [27, 32, 36] used this technique to suggest and analyze some multi step iterative methods for solving various classes of mixed quasi variational inequalities. We would like to mention that a substantial number of numerical methods can be obtained as special cases from this technique.

Noor [25, 26, 34, 36] introduced some new classes of variational inequalities involving an arbitrary operator, which are known as general quasi variational inequalities. It have shown that nonsymmetric, non-positive and odd order obstacle initial (boundary) vale problems can be studied in the unified frame work of general variational inequalities. It is worth mentioning that, if the leading operator in these general variational inequalities is an identity operator, then these general variational inequalities are known as inverse variational inequalities. For the applications, generalizations, numerical methods and other aspects of inverse variational inequalities (inclusions), see [5, 11, 15, 16, 41–43, 52, 53, 56] and the references therein.

In this paper, we explore these ideas and techniques to suggest a class of three step iterative methods for solving general mixed inverse quasi variational inequalities. The convergence of these methods requires only that the operator is partially relaxed strongly monotone and skew symmetry of the bifunction, which is weaker than cocoercive. Our results can be viewed as a significant and novel application of the recent results for solving the general mixed inverse quasi variational inequalities. Since general mixed inverse quasi variational inequalities include inverse quasi variational inequalities and inverse complementarity problems as special cases, our results continue to hold for these problems. We have only considered the theoretical aspects of these results. It is an interesting problem to develop some efficient reliable implementable numerical methods and to consider the comparison with other methods.

2.   Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $\varphi(.,.) : H \times H \to R \cup \{+\infty\}$ be a continuous bifunction function.

For given nonlinear operator $g : H \to H$, consider the problem of finding $u \in H$ such that

$$\langle u, g(v) - g(u) \rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in H. \tag{1}$$

The inequality of type (1) is called the general mixed inverse quasi variational inequality. Variational inequality (1) characterizes the Signorini problem with nonlocal friction. If S is a open bounded domain in R^n with regular boundary ∂S, representing the interior of an elastics body subject to external forces and if a part of the boundary may come into contact with a rigid foundation, then the mixed quasi variational inequality (1) is simply a statement of the virtual work for an elastic body restrained by friction forces, assuming that a nonlocal law friction holds. The strain energy of the body corresponding to an admissible displacement v is $\langle Tv; v-u \rangle$. Thus $\langle Tv; v-u \rangle$ is the work produced by the stresses through strains caused by the virtual displacement $(v-u)$. The friction forces are represented by the bifunction $\varphi(., .)$. For the physical and mathematical formulation of the mixed inverse quasi variational inequalities of type (1), see [6, 8, 9, 13, 16, 18, 27–30, 32, 33, 38, 45, 49, 56] and the references therein.

Special cases

(1) If the bifunction $\varphi(., .)$ is lower semi continuous function, then the problem (1) is equivalent to finding $u \in H$, such that

$$0 \in u + g(v) - g(u) + \partial\varphi(g(u), g(u)), \tag{2}$$

is called the general inverse quasi variational inclusion.
(2) We remark that, if $g \equiv I$, the identity operator, then problem (1) is equivalent to finding $u \in H$ such that

$$\langle u, v - u \rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in H, \tag{3}$$

which is called the mixed inverse quasi variational inequalities.
(3) We note that, if the bifunction $\varphi(., .)$ is the indicator function of a closed convex-valued set $K(u) \subseteq H$, that is,

$$\varphi(u, .) \equiv I_{K(u)}(u) = \begin{cases} 0, & \text{if } u \in K(u) \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (1) is equivalent to finding $u \in K(u)$ such that

$$\langle u, g(v) - g(u) \rangle \geq 0, \qquad \forall v \in K(u), \tag{4}$$

is called the general inverse quasi variational inequality [42].

(4) We note that, if $\varphi(v, u) = \varphi(v), \forall u \in H$, then problem (1) is equivalent to finding $u \in H$ such that

$$\langle u, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \quad \forall v \in H, \tag{5}$$

which is called the general mixed inverse variational inequality.

(5) If φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then the problem (5) is equivalent to finding $u \in K$ such that

$$\langle u, g(v) - g(u) \rangle \geq 0, \qquad \forall v \in K. \tag{6}$$

Inequality of the type (6) is known as the general inverse variational inequality, see [41, 42].

(6)   If $K^* = \{u \in H : \langle u, v \rangle \geq 0, \forall v \in K\}$ is a polar cone of a convex cone K in H, then problem (6) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad u \in K^*, \quad \text{and} \quad \langle u, g(u) \rangle = 0, \tag{7}$$

which is known as the general inverse complementarity problem. We note that if $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem(7) is called the inverse quasi(implicit) complementarity problem.

It is clear that problems (3)-(7) are special cases of the general mixed inverse quasi variational inequality (1). In brief, for a suitable and appropriate choice of the operators T, g, $\varphi(.,.)$ and the space H, on can obtain a wide class of inverse variational inequalities and complementarity problems. This clearly shows that problem (1) is quite general and unifying one. It is an open problem to explore the applications of the problem (1) and its variant forms in various branches of pure and applied sciences such as equilibrium problems, machine learning, data analysis, traffic problems, financial mathematics, risk analysis and optimization.

We also need the following concepts.

Lemma 2.1. $\forall u, v \in H$, we have

$$2\langle u, v \rangle = ||u + v||^2 - ||u||^2 - ||v||^2. \tag{8}$$

Definition 2.1. $\forall u, v, z \in H$, an operator $g : H \to H$ is said to be:

(1) g-**relaxed strongly monotone,** if there exists a constant $\gamma > 0$ such that

$$\langle u - v, g(u) - g(v) \rangle \geq -\gamma ||g(u) - g(v)||^2$$

(2) g-**partially relaxed strongly monotone**, if there exists a constant $\alpha > 0$ such that

$$\langle u - v, g(z) - g(v) \rangle \geq -\alpha ||g(u) - g(z)||^2$$

(3) g-**co-coercive**, if there exists a constant $\mu > 0$ such that

$$\langle u - v, g(u) - g(v) \rangle \geq \mu ||u - v||^2.$$

We remark that, if $z = u$, then g-partially relaxed strongly monotonicity is exactly g-monotonicity of the operator g. It can be shown that g-co-coercivity implies g-partially relaxed strongly monotonicity but the converse is not true. This shows that the concept of partially relaxed strongly monotonicity is weaker than co-coercivity. We also need the following assumption regarding the bifunction $\varphi(.,.)$, which plays a crucial part in the convergence analysis of the new iterative methods.

Assumption 2.1. *The bifunction* $\varphi(.,.) : H \times H \longrightarrow R \cup \{\infty\}$ *is said to be skew-symmetric, if*

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in H. \tag{9}$$

We remark that, if $\varphi(.,.)$ is a bilinear function, then $\varphi(.,.)$ is nonnegative, that is

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) = \varphi(u - v, u - v) \geq 0, \quad \forall u, v \in H.$$

It is worth mentioning that the points $u, u), (u, v), (v, u), (v, v)$ make up a set of the four vertices of the square. In fact, the skew-symmetric bifunction $\varphi(.,.)$ can be written in the

$$\frac{1}{2}\varphi(u, u) + \frac{1}{2}\varphi(v, v) \geq \frac{1}{2}\varphi(u, v) + \frac{1}{2}\varphi(v, u), \quad \forall u, v \in H.$$

This shows that the arithmetic average value of the skew-symmetric bifunction calculated at the north-east and south-west vertices of the square is greater than or equal to the arithmetic average value of the skew-symmetric bifunction computed at the north-west and south-west vertices of the same square. The skew-symmetric bifunction have the properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for the convex functions. For the properties and applications of the skew-symmetric bifunction, see Noor [30, 31].

3. Main Results

In this section, we suggest and analyze a new iterative method for solving the problem (1) by using the auxiliary principle technique of Glowinski et al. [14]. This technique deals with finding the auxiliary variational inequality and proving that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. Glowinski et al. [14] used this technique to study the existence of a solution of mixed variational inequalities. Noor [30–32, 36] and Noor et al.[45, 47, 48] have used this technique to suggest some predictor-corrector methods for solving various classes of variational inequalities and related problems. It is well known [36] that a substantial number of numerical methods can be obtained as special cases from this technique.

For a given $u \in H$ satisfying (1), consider the problem of finding a unique $w \in H$ such that

$$\langle \rho u + g(w) - g(u), g(v) - g(w) \rangle + \rho\varphi(g(v), g(u)) - \rho\varphi(g(u), g(u)) \geq 0, \quad \forall v \in H, \tag{10}$$

where $\rho > 0$ is a constant, is called the auxiliary general mixed inverse quasi variational inequality.

We use the auxiliary problem (10) to discuss the existence of a solution of the problem (1).

Theorem 3.1. *Let the operator g be Lipschitz operator with constant σ. If*

$$\rho < 1 - \sigma, \quad \sigma < 1. \tag{11}$$

then there exists a unique solution of the problem (1).

Proof. We use the auxiliary principle technique to prove the existence of a solution of (1).

It is clear that (10) defines a mapping w connecting the both problems (1) and (10). To prove the existence of a solution of (1), it is enough to show that the mapping w defined by (10) is a contraction mapping. Let $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2$) satisfy the auxiliary problem (10). Then

$$\langle \rho u_1 + g(w_1) - g(u_1), g(v) - g(w_1) \rangle$$
$$+ \rho \varphi(g(v), g(u_1)) - \rho \varphi(g(u_1), g(u_1)) \geq 0, \forall v \in H, \tag{12}$$
$$\langle \rho u_2 + g(w_2) - g(u), g(v) - g(w_2) \rangle$$
$$+ \rho \varphi(g(v), g(u_2)) - \rho \varphi(g(u_2), g(u_2)) \geq 0, \forall v \in H, \tag{13}$$

Taking $v = w_2$ in (12), $v = w_1$ in (13), using skew symmetry of the bifunction $\varphi(.,.)$ and adding the resultant, we have

$$\begin{aligned}
\|g(w_1) - g(w_2)\|^2 &= \langle g(w_1) - g(w_2), g(w_1) - g(w_2) \rangle \\
&= \langle g(u_1) - g(u_2) - \rho(u_1 - u_2), g(w_1) - g(w_2) \rangle \\
&\leq \|g(u_1) - g(u_2) - \rho(u_1 - u_2)\| \|g(w_1) - g(w_2)\| \tag{14}
\end{aligned}$$

From (14), we have

$$\|g(w_1) - g(w_2)\|^2 \leq \|g(u_1) - g(u_2) - \rho(u_1 - u_2)\| \|g(w_1) - g(w_2)\|,$$

which implies, using the Lipschitz continuity of the operator A with constant $\sigma > 0$, that

$$\begin{aligned}
\|g(w_1) - g(w_2)\| &\leq \|g(u_1) - g(u_2) - \rho(u_1 - u_2)\| \leq \|g(u_1) - g(u_2)\| + \rho \|u_1 - u_2\| \\
&\leq (\sigma + \rho) \|u_1 - u_2\| = \theta \|u_1 - u_2\|,
\end{aligned}$$

where

$$\theta = \sigma + \rho.$$

Since $\theta < 1$, from (12), it follows

$$\|g(w_1) - g(w_2)\| \leq 0.$$

Thus

$$\lim_{n \to \infty} g(w_1) = \lim_{n \to \infty} g(w_2) \Rightarrow w_1 = w_2,$$

This shows mapping w defined by the relation (10)is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ satisfying the problem (1).

$\square$

We note that, if $w = u$, then clearly w is a solution of the general mixed inverse quasi variational inequality (1). This observation enables us to suggest the following iterative method for solving the general mixed inverse quasi variational inequalities (1).

Algorithm 1. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho w_n + g(u_{n+1}) - g(w_n) \quad, \quad g(v) - g(u_{n+1}) \rangle + \rho \varphi(g(v), g(u_{n+1}))$$
$$-\rho \varphi(g(u_{n+1}), g(u_{n+1})) \geq 0, \quad \forall v \in H \qquad (15)$$

$$\langle \beta y_n + g(w_n) - g(y_n) \quad, \quad g(v) - g(w_n) \rangle + \beta \varphi(g(v), g(w_n))$$
$$-\beta \varphi(g(w_n), g(w_n)) \geq 0, \quad \forall v \in H, \qquad (16)$$

and

$$\langle \eta u_n + g(y_n) - g(u_n) \quad, \quad g(v) - g(y_n) \rangle + \mu \varphi(g(v), g(y_n))$$
$$-\mu \varphi(g(y_n), g(y_n)) \geq 0, \quad \forall v \in H. \qquad (17)$$

where $\rho > 0$, $\eta > 0$ and $\beta > 0$ are constants.

Note that if $g \equiv I$, the identity operator, then Algorithm 1 reduces to:

Algorithm 2. *For a given u_0, compute u_{n+1} by the iterative scheme*

$$\langle \rho w_n + u_{n+1} - w_n, v - u_{n+1} \rangle + \rho \varphi(v, u_{n+1}) - \rho \varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in H,$$

$$\langle \beta y_n + w_n - y_n, v - w_n \rangle + \beta \varphi(v, w_n) - \beta \varphi(w_n, w_n) \geq 0, \quad \forall v \in H$$

and

$$\langle \eta u_n + y_n - u_n, v - y_n \rangle + \mu \varphi(v, y_n) - \mu \varphi(y_n, y_n) \geq 0, \quad \forall v \in H.$$

Algorithm 2 appears to be a new one for solving mixed inverse quasi variational inequalities (3).

If $\varphi(v, u)$ is a proper, convex and lower-semicontinuous differentiable function with respect to the first argument, then Algorithm 1 collapses to:

Algorithm 3. *For a given u_0, compute u_{n+1} by the iterative scheme*

$$\begin{aligned}
g(u_{n+1}) &= J_{\varphi(u_{n+1})}[g(w_n) - \rho w_n], \\
g(w_n) &= J_{\varphi(w_n)}[g(y_n) - \beta y_n], \\
g(y_n) &= J_{\varphi(y_n)}[g(u_n) - \eta u_n], \quad n = 0, 1, 2 \ldots,
\end{aligned}$$

where $J_{\varphi(u)} \equiv (I + \rho \partial \varphi(.,u))^{-1}$ is the resolvent operator associated with the subdifferential $\partial \varphi(.,u)$ of the bifunction $\varphi(.,.)$, see [19].

If the function $\varphi(.,u), \quad \forall u \in H$, is the indicator function of a closed convex-valued set $K(u) \subseteq H$, then Algorithm 1 reduces to the following method for solving general inverse quasi variational inequalities (6).

Algorithm 4. *For a given u_0, compute u_{n+1} by the iterative schemes*

$$\langle \rho w_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1}) \rangle \geq 0, \quad \forall v \in K(u)$$

$$\langle \beta y_n + g(w_n) - g(y_n), g(v) - g(w_n) \rangle \geq 0, \quad \forall v \in K(u)$$

and

$$\langle \eta u_n + g(y_n) - g(u_n), g(v) - g(y_n) \rangle \geq 0, \quad \forall v \in K(u)$$

which is equivalent to:

Algorithm 5. *For a given u_0, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} g(u_{n+1}) &= \Pi_{K(u_{n+1})}[g(w_n) - \rho w_n], \\ g(w_n) &= \Pi_{K(w_n)}[g(y_n) - \beta y_n], \\ g(y_n) &= \Pi_{K(y_n)}[g(u_n) - \eta u_n], \quad n = 0, 1, 2 \ldots, \end{aligned}$$

where $J_{\varphi(u)} \equiv (I + \rho \partial \varphi(.,u))^{-1}$ is the resolvent operator associated with the subdifferential $\partial \varphi(.,u)$ of the bifunction $\varphi(.,.)$.

For $K(u) = K$, closed convex set, Algorithm 4 and Algorithm 5 reduce the following algorithm for solving the general inverse variational inequalities.

Algorithm 6. *For a given u_0, compute u_{n+1} by the iterative schemes*

$$\langle \rho w_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1}) \rangle \geq 0, \quad \forall v \in K$$

$$\langle \beta y_n + g(w_n) - g(y_n), g(v) - g(w_n) \rangle \geq 0, \quad \forall v \in K$$

and

$$\langle \eta u_n + g(y_n) - g(u_n), g(v) - g(y_n) \rangle \geq 0, \quad \forall v \in K.$$

which is equivalent to

Algorithm 7. *For a given u_0, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} g(u_{n+1}) &= \Pi_K[g(w_n) - \rho w_n], \\ g(w_n) &= \Pi_K[g(y_n) - \beta y_n], \\ g(y_n) &= \Pi_K[g(u_n) - \eta u_n], \quad n = 0, 1, 2 \ldots, \end{aligned}$$

where Π_K is the implicit projection of H onto the closed convex set K.

For a suitable choice of the operators and the space H, one can obtain various new and known methods for solving variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 1, we need the following result.

Lemma 3.1. *Let $\bar{u} \in H$ be the exact solution of (1) and u_{n+1} be the approximate solution obtained from Algorithm 1. If the operator $g : H \longrightarrow H$ is a g- partially relaxed strongly monotone operator with constant $\alpha > 0$, and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(w_n) - g(\bar{u})\|^2 - (1 - 2\alpha\rho)\|g(u_{n+1}) - g(w_n)\|^2 \quad (18)$$

$$\|g(w_n) - g(\bar{u})\|^2 \leq \|g(y_n) - g(\bar{u})\|^2 - (1 - 2\alpha\beta)\|g(w_n) - g(y_n)\|^2 \quad (19)$$

$$\|g(y_n) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - (1 - 2\alpha\mu)\|g(y_n) - g(u_n)\|^2. \quad (20)$$

Proof. Let $\bar{u} \in H$ be solution of (1). Then

$$\langle \rho\bar{u}, g(v) - g(\bar{u})\rangle + \rho\varphi(g(v), g(\bar{u})) - \rho\varphi(g(\bar{u}), g(\bar{u})) \geq 0, \quad \forall v \in H, \quad (21)$$

$$\langle \beta\bar{u}, g(v) - g(\bar{u})\rangle + \beta\varphi(g(v), g(\bar{u})) - \beta\varphi(g(\bar{u}), g(\bar{u})) \geq 0, \quad \forall v \in H, \quad (22)$$

and

$$\langle \eta\bar{u}, g(v) - g(\bar{u})\rangle + \mu\varphi(g(v), g(\bar{u})) - \mu\varphi(g(\bar{u}), g(\bar{u})) \geq 0, \quad \forall v \in H, \quad (23)$$

where $\rho > 0$, $\beta > 0$ and $\eta > 0$ are constants.

Now taking $v = u_{n+1}$ in (21) and $v = \bar{u}$ in (12), we have

$$\langle \rho T\bar{u}, g(u_{n+1}) - g(\bar{u})\rangle + \rho\varphi(g(u_{n+1}), g(\bar{u})) - \rho\varphi(g(\bar{u}), g(\bar{u})) \geq 0 \quad (24)$$

and

$$\langle \rho Tw_n + g(u_{n+1}) - g(w_n), g(\bar{u}) - g(u_{n+1})\rangle + \rho\varphi(g(\bar{u}), g(u_{n+1}))$$
$$- \rho\varphi(g(u_{n+1}), g(u_{n+1})) \geq 0. \quad (25)$$

Adding (24) and (25), we have

$$
\begin{aligned}
&\langle g(u_{n+1}) - g(w_n), g(\bar{u}) - g(u_{n+1})\rangle \\
&\geq \rho\langle w_n - \bar{u}, g(u_{n+1}) - g(\bar{u})\rangle \\
&+ \varphi(g(\bar{u}), g(\bar{u})) - \varphi(g(\bar{u}), g(u_{n+1})) - \varphi(g(u_{n+1}), g(\bar{u})) \\
&+ \varphi(g(u_{n+1}), g(u_{n+1})) \\
&\geq -\alpha\rho\|g(u_{n+1}) - g(w_n)\|^2, \quad (26)
\end{aligned}
$$

where we have used the fact that g is g-partially relaxed strongly monotone with constant $\alpha > 0$ and the skew-symmetry of the bifunction $\varphi(.,.)$.

Setting $u = g(\bar{u}) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(w_n)$ in (9), we obtain

$$\langle g(u_{n+1}) - g(w_n), g(\bar{u}) - g(u_{n+1}) \rangle = \frac{1}{2}\{\|g(\bar{u}) - g(w_n)\|^2 - \|g(\bar{u}) - g(u_{n+1})\|^2 \\ - \|g(u_{n+1}) - g(w_n)\|^2.\} \tag{27}$$

Combining (25) and (27), we have

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(w_n) - g(\bar{u})\|^2 - (1 - 2\alpha\rho)\|g(u_{n+1}) - g(w_n)\|^2,$$

the required (18).

Taking $v = \bar{u}$ in (20) and $v = w_n$ in (22), we have

$$\langle \beta\bar{u}, g(w_n) - g(\bar{u}) \rangle + \beta\varphi(g(w_n), g(\bar{u})) - \beta\varphi(g(\bar{u}), g(\bar{u})) \geq 0 \tag{28}$$

and

$$\langle \beta y_n + g(w_n) - g(y_n), g(\bar{u}) - g(w_n) \rangle + \beta\varphi(g(\bar{u}), g(w_n)) \\ - \beta\varphi(g(w_n), g(w_n)) \geq 0. \tag{29}$$

Adding (28) and (29) and rearranging the terms, we have

$$\langle g(w_n) - g(y_n) , g(\bar{u}) - g(w_n) \rangle \geq \beta\langle Ty_n - T\bar{u}, g(w_n) - g(\bar{u}) \rangle + \varphi(g(\bar{u}), g(\bar{u})) \\ - \varphi(g(\bar{u}), g(w_n)) - \varphi(g(w_n), g(\bar{u})) + \varphi(g(w_n), g(w_n)) \\ \geq -\beta\alpha\|g(y_n) - g(w_n)\|^2, \tag{30}$$

since g is a g-partially relaxed strongly monotone operator with constant $\alpha > 0$ and $\varphi(.,.)$ is a skew-symmetry.

Now taking $v = g(w_n) - g(y_n)$ and $u = g(\bar{u}) - g(w_n)$ in (9), (30) can be written as

$$\|g(\bar{u}) - g(w_n)\|^2 \leq \|g(\bar{u}) - g(y_n)\|^2 - (1 - 2\beta\alpha)\|g(y_n) - g(w_n)\|^2,$$

the required (19).

Similarly, by taking $v = \bar{u}$ in (20) and $v = u_{n+1}$ in (25) and using the g-partially relaxed strongly monotonicity of the operator T, and using the skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\langle g(y_n) - g(u_n), g(\bar{u}) - g(y_n) \rangle \geq -\eta\alpha\|g(y_n) - g(u_n)\|^2. \tag{31}$$

Letting $v = y_n - u_n$, and $u = \bar{u} - y_n$ in (9), and combining the resultant with (31), we have

$$\|g(y_n) - g(\bar{u})\|^2 \leq \|g(\bar{u}) - g(u_n)\|^2 - (1 - 2\eta\alpha)\|g(y_n) - g(u_n)\|^2,$$

the required (20).   □

Theorem 3.2. *Let $\bar{u} \in H$ be a solution of (1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If H is a finite dimensional space, g is an injective map and $0 < \rho < 1/2\alpha, 0 < \beta < 1/2\alpha, 0 < \eta < 1/2\alpha$, then $\lim_{n\to\infty} u_n = \bar{u}$.*

Proof. Let $\bar{u} \in H$ be a solution of (1). Then, from (18), (19) and (20), it follows that the sequences $\{\|g(u_n) - g(\bar{u})\|\}$, $\{\|g(y_n) - g(\bar{u})\|\}$ and $\{\|g(w_n) - g(\bar{u})\|\}$ are nonincreasing and consequently the sequences $\{w_n\}, \{y_n\}$ and $\{u_n\}$ are bounded and

$$\sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|g(u_{n+1}) - g(u_n)\|^2 \leq \|g(w_0) - g(\bar{u})\|^2$$

$$\sum_{n=0}^{\infty}(1 - 2\alpha\beta)\|g(w_n) - g(u_n)\|^2 \leq \|g(y_0) - g(\bar{u})\|^2$$

$$\sum_{n=0}^{\infty}(1 - 2\alpha\mu)\|g(y_n) - g(u_n)\|^2 \leq \|g(u_0) - g(\bar{u})\|^2,$$

which implies that

$$\lim_{n \to \infty} \|g(u_{n+1}) - g(w_n)\| = 0$$
$$\lim_{n \to \infty} \|g(w_n) - g(y_n)\| = 0$$
$$\lim_{n \to \infty} \|g(y_n) - g(u_n)\| = 0.$$

Thus

$$\lim_{n \to \infty} \|g(u_{n+1}) - g(u_n)\| = \lim_{n \to \infty} \|g(u_{n+1}) - g(w_n)\| + \lim_{n \to \infty} \|g(w_n) - g(y_n)\|$$
$$+ \lim_{n \to \infty} \|g(y_n) - g(u_n)\| = 0. \tag{32}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n and y_n by u_{n_j} in (18),(19) and (20), taking the limit $n_j \longrightarrow \infty$ and using (32), we have

$$\langle T\hat{u}, g(v) - g(\hat{u})\rangle + \varphi(g(v), g(\hat{u}))) - \varphi(g(\hat{u}), g(\hat{u})) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the general mixed inverse quasi variational inequality (1) and

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n \to \infty} g(u_n) = g(\hat{u}).$$

Since g is invertible, so

$$\lim_{n \to \infty} (u_n) = \hat{u},$$

the required result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Using the technique and ideas of this paper, one can consider the convergence criteria of other Algorithms with suitable and appropriate modifications.

Conclusion: In this paper, we have considered and studied a new class of general mixed inverse quasi variational inequalities involving some arbitrary different operators. The auxiliary principle technique is used to suggest and propose some three step iterative for solving these inverse quasi variational inequalities. Making use of the skew symmetry of the involved bifunction $\varphi(.,.)$, convergence criteria of the proposed iterative method is considered. Several important special cases are also pointed out. We would like to point out that very few numerical methods are available for solving general mixed inverse quasi variational inequalities. In spite of these activities, further efforts are needed to develop numerical implementable methods. We expect that the ideas and techniques of this paper will motivate and inspire the interested readers to explore the applications of general inverse quasi variational inequalities in various fields such as asymptotic control, equilibrium, traffic equilibrium problem, designing solar panels, sensing, medical sciences, finance, and image in painting .

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References

[1] Alshejary, A. A., Noor, M. A., Noor, K. I.,2024. Recent developments in general quasi variational inequalities, Inter. J. Anal. Appl. 22, $DOI : https : //doi.org/10.28924/2291 - 8639 - 22 - 2024 - 84$.

[2] Ashish, K., Rani, M., Chugh, R., 2014. Julia sets and Mandelbrot sets in Noor orbit, Appl. Math. Comput., 228(1), 615-631.

[3] Ashish, K., Chugh, R., Rani,M., 2021. Fractals and Chaos in Noor Orbit: a four-step feedback approach, Lap Lambert Academic Publishing, Saarbrucken, Germany.

[4] Ashish, K., Cao, J., Noor, M. A., 2023. Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia, J. Appl. Anal. Computation, 13(5), $2452\text{-}2470.doi : 10.11948/20220350 http : //www.jaac - online.com$

[5] Aussel, D., Gupta, R., Mehra, A., 2013. Gap functions and error bounds for inverse quasi-variational inequality problems, J. Math. Anal. Appl. 407(2), 270-280.

[6] Baiocchi, C., Capelo, A., 1984. Variational and Quasi Variational Inequalities, J. Wiley and Sons, New York, London.

[7] Barbagallo, A., Lo Bainco, S. G., 2024. A random elastic traffic equilibrium problem via stochastic quasi-variational inequalities, Commun. Nonlin. Sci. Numer. Simul. 131,1-11, https://doi.org/10.1016/j.cnsns.2023.107798

[8] Bensoussan, A., Lions, J. L., 1978. and J. L. Lions, Application des Inequalities Variationnelles en Control eten Stochastique, Dunod, Paris.

[9] Brezis, H., 1973. Operateurs Maximaux Monotone et Semigroups de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam.

[10] R. W. Cottle, R. W., Pang, Jong-S., Stone, R. E., 1992. The Linear Complementarity Problem, SIAM Publication, Philadelphis, USA.

[11] Dey, S., Reich, S., (2023). A dynamical system for solving inverse quasi-variational inequalities, Optimization,72., DOI: 10.1080/02331934.2023.21735

[12] Dixit, A., Sahu, D.R., Gautam, P., Som, T., Yao, J. C., 2021. An accelerated forward-backward splitting algorithm for solving inclusion problems with applications to regression and link prediction problems, J. Nonlin. Var, Anal. 5, 79-101.

[13] Giannessi, F., Maggeri, A., Pardalos Maugeri, P. M., 2001. Equilibrium Problems: Non-smooth Optimization and Variational Inequality Models, Kluwer Academic Publishers, Dordrecht, Holland.

[14] Glowinski, R., Lions, J. L., Tremolieres, R., 1981. Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.

[15] Han, Y., Huang, N., Lu, J., Xiao, Y., 2017. Existence and stability of solutions to inverse variational inequality problems. Appl. Math. Mech(English Edition),. 38(5), 749-764. doi: 10.1007/s10483-017 2191-9

[16] He, B., He, X., Liu, H. X., 2010. Solving a class of constrained 'black-box' inverse variational inequalities. Eur. J. Oper. Res. 204, 391-401.

[17] Jabeen, S., Noor, M. A., Noor, K. I., 2021. Some new inertial projection methods for quasi variational Inequalities, Appl. Math. E-Notes, 2021, 172-178.

[18] Kikuchi, N., Oden, J. T., 1988. Contact problems in Elasticity, SIAM Publishing Co., Philadelphia, NJ. USA.

[19] Kinderlehrer. D., Stampacchia, G., 2000. An Introduction to Variational Inequalities and Their Applications, SIAM Publishing Co., Philadelphia, NJ, USA.

[20] Lions, J. L., Stampacchia,G., 1967. Variational inequalities, Commun. Pure Appl. Math. 20, 493-519.

[21] Natarajan, S. K., Negi, D., 2024. Green innovations utiling fractal and power for solar panel optimization, in Green Innovations for Industrial Development and Business Sustainability (R.Sharma, G. Rana and S. Agarwal (Eds)), CRC Press, Florida, Boca Raton, USA, pp.146-152.

[22] Noor, M. A., 1975. On Variational Inequalities, Ph.D Thesis, Brunel University, London, UK.

[23] Noor, M. A., 1987. On a class of variational inequalities, J. Math. Anal. Appl. 128, 138-155.

[24] Noor, M. A., 1988. Nonlinear variational inequalities in elastostatics, Inter. J. Eng. Sci. 26, 1043-1053.

[25] Noor, M. A., 1988. General variational inequalities, Appl. Math. Letters, 1, 119-121.

[26] Noor, M. A., 1988. Quasi variational inequalities, Appl. Math. Letters, 1(4),367-370.

[27] Noor, M. A., 1999. Some algorithms for general monotone mixed variational inequalities, Math. Computer Modelling, 29(7),1-9.

[28] Noor, M. A., 2001. Some predictor-corrector algorithms for multivalued variational inequalities, J. Optim. Theory Appl. 108, 659-670.

[29] Noor, M. A., 2002. Proximal methods for mixed quasi variational inequalities. J. Optim. Theory Appl. 115, 453-459, https://doi.org/10.1023/A:1020800608324

[30] Noor, M. A., 2003. Mixed quasi variational inequalities, Appl. Math. Compout. 146(2-3), 553-578. https://doi.org/10.1016/S0096-3003(02)00605-7

[31] Noor, M. A., 2003. Iterative methods for general mixed quasi variational inequalities, J. Optim. Theory Appl., 119, 123-136. https://doi.org/10.1023/B:JOTA.0000005044.11591.bd

[32] Noor, M. A., 2004. Fundamentals of mixed quasi variational inequalities, Inter. J. Pure Appl. Math. 15, 137-250.

[33] Noor, M. A., 1998. Generalized set-valued variational inclusions and resolvent equations, J. Math. Anal. Appl. 228,206-220.

[34] Noor, M. A., 2000. New approximation schemes for general variational inequalities, J. Math. Aanal. Appl. 251, 217-229.

[35] Noor, M. A., 2001. Three-step iterative algorithms for multivalued quasi variational inclusions, J. Math. Anal. Appl. 255, 589-604.

[36] Noor, M. A., 2004. Some developments in general variational inequalities, Appl. Math. Comput., 152, 199-277.

[37] Noor, M. A., 2009. Extended general variational inequalities, Appl. Math. Lett., 22, 182-185.

[38] Noor, M. A., Noor, K. I., 2022. New Inertial approximation schemes for general quasi variational inclusions, Filomat, 36(18), 6071-6084.

[39] Noor, M. A., Noor, K. I., 2024. New classes of general quasi variational inclusions, J. Advan. Math. Stud., 17(1), 93-105.

[40] Noor, M. A., Noor, K. I., 2024. Some novel aspects and applications of Noor iterations and Noor orbits, J. Advan. Math. Stud., 17(3), 276-284.

[41] Noor, M. A., Noor, K. I., 2025. New iterative methods for inverse variational inequalities, J. Advan. Math. Stud., 18(4), in Press.

[42] Noor, M. A., Noor, K. I., 2025. New iterative methods and sensitivity analysis for inverse quasi variational inequalities, Earthhline J. Math. Sci., 15(4), 495-539.

[43] Noor, M. A., Noor, K. I., 2025. Iterative methods for solving general inverse quasi variational inclusions, Preprint.

[44] M. A. Noor, M. A., Oettli, W., 1994. On general nonlinear complementarity problems and quasi equilibria, Le Mathemath(Catania). 49, 313-331.

[45] Noor, M. A., Noor, K. I., Al-Khaled, K., 2004. Predictor-Corrector methods for general mixed quasi variational inequalities, Appl. Math. Comput. 157(3), 643-652, $https://doi.org/10.1016/j.amc.2003.08.060$

[46] Noor, M. A., Noor, K. I., Rassias, Th. M., 1988. Set-valued resolvent equations and mixed variational inequalities, J. Math. Anal. Appl. , 741-759.

[47] Noor, M. A., Noor, K. I., Rassias, M. Th., 2020. New trends in general variational inequalities, Acta Appl. Mathematica, 170(1), 981-1064.

[48] Noor, M. A., Noor, K. I., Rassias, Th. M., 1993. Some aspects of variational inequalities, J. Comput. Appl. Math. 47, 285-312.

[49] Noor, M. A., Noor, K. I., Treanta, S., Nonlaopon, K., 2022. On three-step iterative schemes associated with general quasi-variational inclusions, Alexandria Eng. J., 61,12051-12059.

[50] Stampacchia, G., 1964. Formes bilineaires coercivites sur les ensembles convexes, C. R. Acad. Sci.Paris., 258, 4413-4416.

[51] Suantai, S., Noor, M. A., Kankam, K., Cholamjiak, P., 2001. Novel forward-backward algorithms for optimization and applications to compressive sensing and image inpainting, Advan. Differ. Eqs. 2021, Id-265, DOI: 10.1186/s13662-021-03422-9.

[52] T. Q. Trinh, T. Q., Vuong, P. T., 2024. The projection algorithm for inverse quasi-variational inequalities with applications to traffic assignment and network equilibrium control problems, Optimization, 2024, 1-25. https://doi.org/10.1080/02331934.2024.2329788

[53] Vuong. P. T., He, X., Thong, D. V., 2021. Global exponential stability of a neural network for inverse variational inequalities. J. Optim. Theory Appl. 190, 915-930. $doi: 10.1007/s10957 - 021 - 01915 - x$

[54] H. K. Xu, H. K., 2002. Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. 66, 240-256. .

[55] A. Yadav, A., Jha, K., 2016. Parrondo's paradox in the Noor logistic map, Int. J. Adv. Research Eng. Technology, 7(5), 01-06.

[56] Zhang, Y., Yu, G., 2022. Error bounds for inverse mixed quasi-variational inequality via generalized residual gap functions, Asia-Pacific J. Operational Research, 39(2), ID= 2150017 https://doi.org/10.1142/S0217595921500172.